Angle of arrival estimation using sparse linear arrays for the superior case

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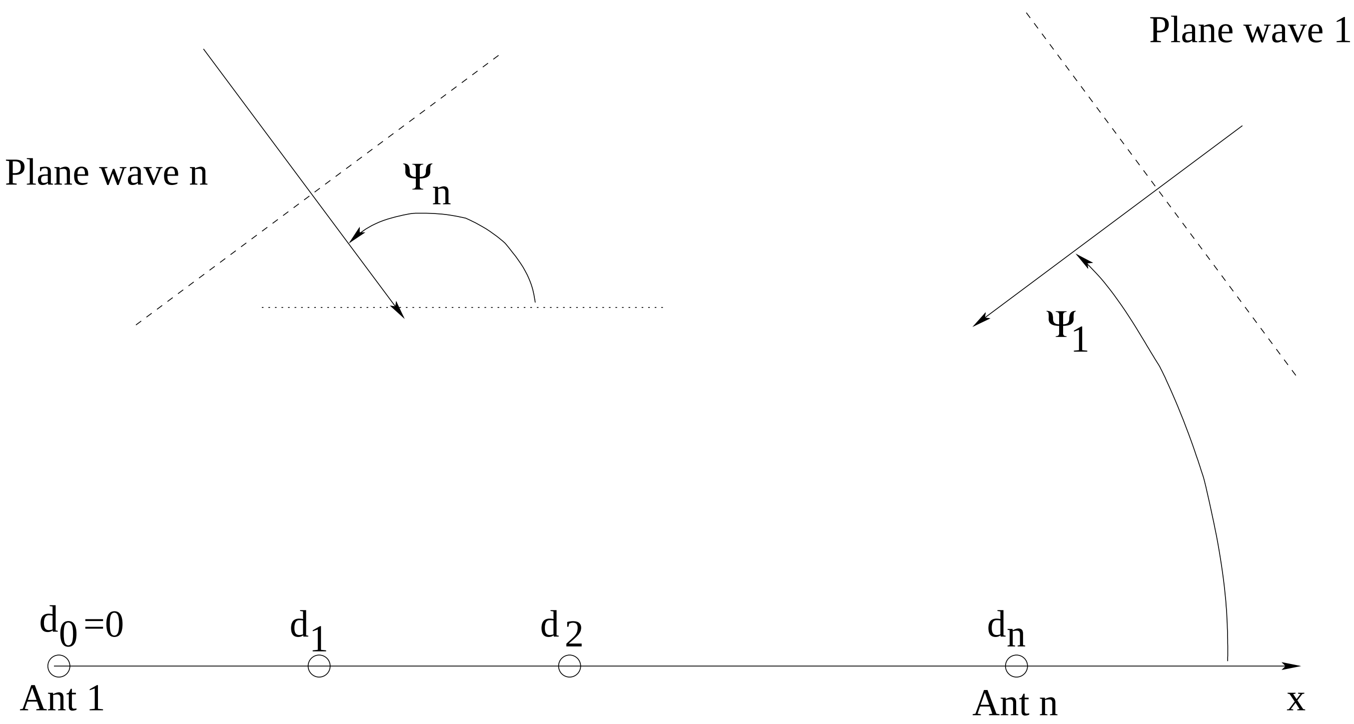
Outline of presentation

- Introduction to Angle of Arrival (AoA) problem
- Cramer-Rao bounds for a class of non-linear estimators: Subspace projection methods
- Oversampling based AoA estimation
- Introduction to Iterative Bayesian Methods
- Numerical results
- Conclusions

Introduction to Angle of Arrival (AoA) estimation problem

The problem: Estimate the Angle of Arrival (AOA) for multiple plane waves with arbitrary frequency using a linear array.



Angle of Arrival (AoA) problem: Continued

$$\mathbf{y}(t) = \mathbf{A}(\theta)\mathbf{x}(t) + \mathbf{e}(t) \quad \forall \;\; t = 1, \cdots, T. \tag{1}$$

t indicates time (a snapshot), and $\mathbf{y}$ is a column vector of M elements corresponding to M array sensors, while $\mathbf{x}(t)$ is a column vector with N elements (the number of sources). $\mathbf{A}$ is the direction matrix with dimentions (M, N) and $\mathbf{e}$ is an additive noise. θ denotes N angles $\theta_i \; \forall \; i \in \{1, \cdots, N\}$ corresponding to the N sources and are to be estimated as the AoA parameters of interest.

Angle of Arrival (AoA) problem: Continued

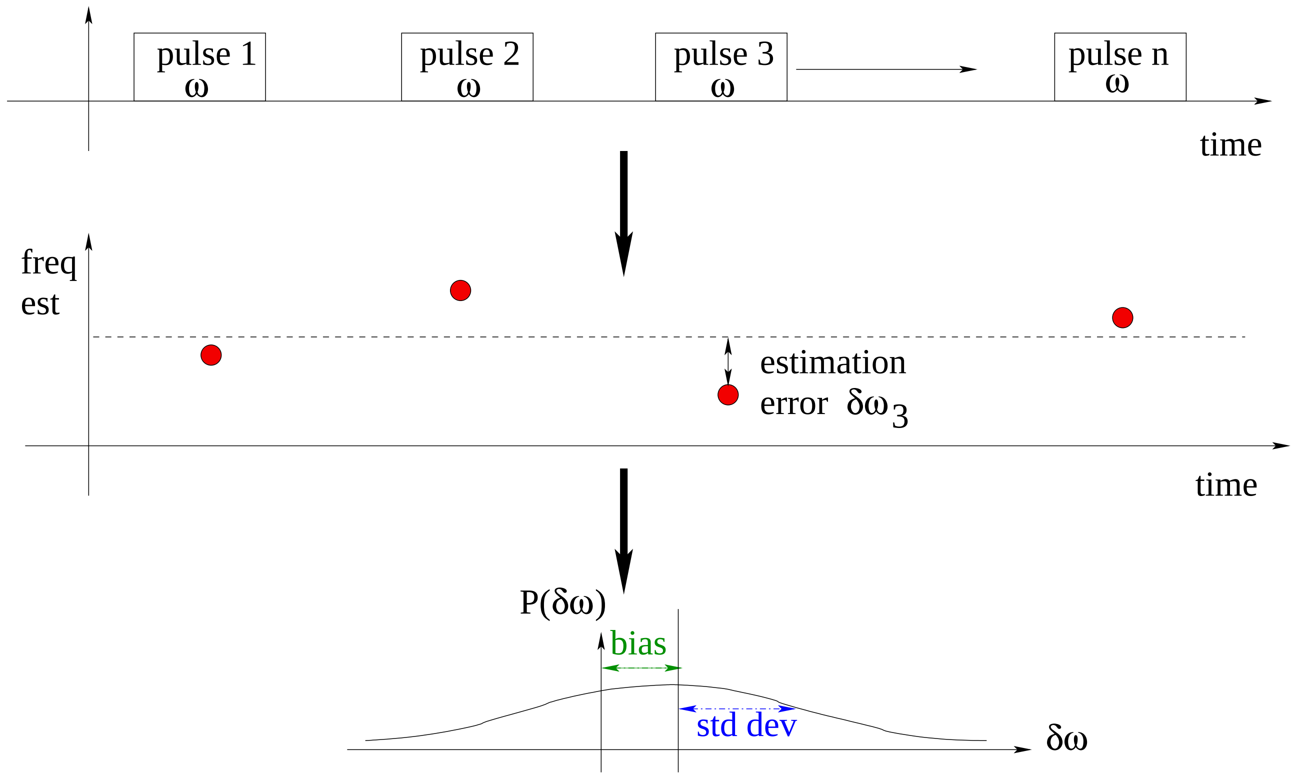
In MUSIC a number of assumptions on the noise $\mathbf{e}$ and signal $\mathbf{x}(t)$ statistis are made.
The covariance matrix of the received signal is denoted by $\mathbf{R}$ and is given by

$$\mathbf{R} = E\{\mathbf{y}(t)\mathbf{y}(t)^{\dagger}\} = \mathbf{A}(\theta)E\{\mathbf{x}(t)\mathbf{x}(t)^{\dagger}\}\mathbf{A}(\theta)^{\dagger} + \sigma^2\mathbf{I}. \quad (2)$$

Here † implies the conjugate transpose and $E\{\}$ the expectation operator, while σ is the standard deviation of the assumed additive white Gaussian noise. $\mathbf{I}$ denotes an identity matrix.

Cramer-Rao bounds for a class of non-linear estimators

How do we measure the performance of an estimator?



Cramer-Rao bounds for a class of non-linear estimators: Continued

Theorem: (Cramer-Rao-Fisher) *Given the SNR and the size of the observation set the variance (or standard deviation) of the estimate of* θ *yielded by any unbiased estimator is at least as high as the inverse of the Fisher information* $I(\theta)$.

$$var(\widehat{\theta}) \geqslant \frac{1}{I(\theta)}$$

Let θ distributed according to probability density function $f(x;\theta)$, x denote the measurements. Then

$$I(\theta) = E\left[\left(\frac{\partial log f(x,\theta)}{\partial \theta}\right)^2\right]$$

Cramer-Rao bounds for a class of non-linear estimators: Continued

A large class of estimation problems can be cast as

$$y = A(\theta)x(t) + e(t) \quad \forall \;\; t = 1, \cdots, T$$

y is a column vector of m elements, $x(t)$ is a column vector with n elements, $A(\theta)$ is a (m, n) matrix and e is a column vector of additive noise components with known distribution.

- Estimating the carrier frequency of a pulse can be written in this form
- Estimating the angle of arrival (AOA) of plane waves on an array can be written in this form

Cramer-Rao bounds for a class of non-linear estimators: Continued

The CRB for this class of estimation problems can be shown to be given by (see Petre Stoica 1989)

$$CRB(\theta) = \frac{\sigma^2}{2}\left\{\sum_{t=1}^{N} Re\{X^\dagger(t)D^\dagger[I - A(A^\dagger A)^{-1}A^\dagger)]DX(t)\}\right\}^2$$

$$X(t) = \begin{bmatrix} x_1(t) & 0 & \cdots & 0 \\ 0 & x_2(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & x_n(t) \end{bmatrix}$$

$$D = [\frac{\partial a_1(\omega_1)}{\partial \omega_1}, \cdots, \frac{\partial a_n(\omega_n)}{\partial \omega_n}]$$

where a_i is the i'th column of A and θ has elements ω.

Difficulties for the MUSIC method

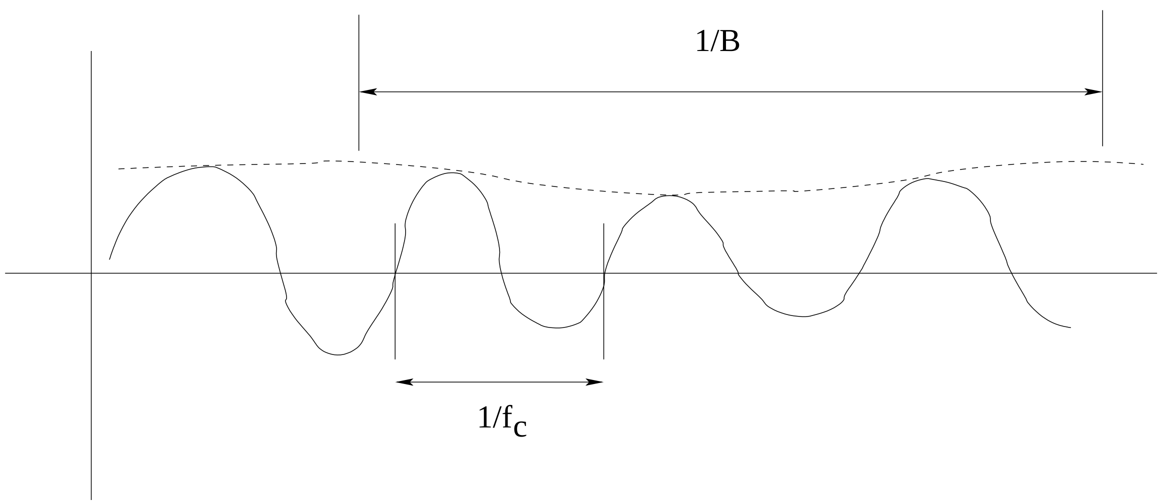
- The CRB is infinite if sources $\geqslant$ than sensors (superior case)
- The AoA estimate is ambiguous if array is *sparse*
- the CRB is large if angles are close to each other

AoA estimation using oversampling and Bayesian methods

Signal bandwidth = B Hz

Sampling frequency = f Hz

$B \ll f$



Piecewise constant formulation

For δt the amplitude over $t \in \{t_0 \cdots, t_0 + \delta t\}$ will be constant. There are say Q samples at a sample period of T seconds.

$$\begin{aligned} y_1(t) &= \alpha_1^1 \exp(j\omega_1 t) + \cdots + \alpha_1^N \exp(j\omega_N t) + e_1(t) \\ y_2(t) &= \alpha_2^1 \exp(j\omega_1 t) + \cdots + \alpha_2^N \exp(j\omega_N t) + e_2(t) \\ &\vdots \\ y_M(t) &= \alpha_M^1 \exp(j\omega_1 t) + \cdots + \alpha_M^N \exp(j\omega_N t) + e_M(t) \qquad (3) \end{aligned}$$

$\mathbf{y}$ is the observed data (a column vector with Q samples), $t \in \{t_0 \cdots, t_0 + \delta t\}$ is time (snapshots), $\mathbf{e}$ noise and $j = \sqrt{(-1)}$.

Piecewise constant formulation: continued

Given the source frequencies were estimated, the complex amplitudes for sersor i is

$$[\alpha_i^1, \cdots, \alpha_i^N]^T = (\mathbf{E}^\dagger \mathbf{E})^{-1} \mathbf{E}^\dagger [y_i(t_0), \cdots, y_i(t_0 + \delta t)]^T \tag{4}$$

where † indicates the Hermitian transpose and ^{T} the transpose,

$$\mathbf{E} = [\mathbf{e}_1, \cdots, \mathbf{e}_N] \tag{5}$$

and

$$\mathbf{e}_k = [1, \exp(j2\pi f_k), \cdots, \exp(\frac{j2\pi f_k}{Q-1})]^T. \tag{6}$$

Time varying formulation

It is possible to estimate α as a function of time.

At each time sample, after α were estimated then the angle can be estimated.

It appears iterative Bayesian Methods are well suited for this approach.

Introduction to Iterative Bayesian Methods

- $\mathbf{x}_k$ is a state vector at time k containing variables that are not directly observable in the digital receiver.
- The *Process model* is denoted by f_k, and any process uncertainty by process noise v_k. Assume Markov model order 1.

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{v}_{k-1})$$

- $\mathbf{f}$ is assumed to be a known but possibly nonlinear function (weak assumption)
- *Objective* The receiver is to estimate $\mathbf{x}_k$ based on a series of measurements up to time k that are noisy, and denoted by $\mathbf{z}_{k=1,2,3,\cdots,k}$

Introduction to Iterative Bayesian Methods: Continued

- Relationship between x and z is given by $\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{w}_k)$
- $\mathbf{h}$ is known and nonlinear, $\mathbf{w}$ is additive (measurement) noise
- In Bayesian inference, we wish to compute some degree of belief in the state x_k given data Z_k up to time k.
- Denote by $\mathbf{Z}$ the entire observed vector sequence up to time k as $\mathbf{Z}_k = \{\mathbf{z}_i, \forall i = 1, \cdots, k\}$
- Assume $p(\mathbf{x}_k|\mathbf{Z}_{k-1}, \mathbf{x}_{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$ i.e. we are dealing with a Markov process of order one
- At time k $p(\mathbf{x}_{k-1}|\mathbf{Z}_{k-1})$ is known $\{p(x_0|z_0)$ is prior at $k = 1$ where z_0 is the set of zero measurements$\}$

Introduction to Iterative Bayesian Methods: Continued

- Applying Bayes' rule we have the **update relation**:

$$p(\mathbf{x}_k|\mathbf{Z}_k) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Z}_{k-1})}{p(\mathbf{z}_k|\mathbf{Z}_{k-1})}$$

- Normalizing constant given is by $p(\mathbf{z}_k|\mathbf{Z}_{k-1}) = \int p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Z}_{k-1})d\mathbf{x}_k$

- Chapman-Kolmogorov equation **predicts** $p(\mathbf{x}_k|\mathbf{Z}_{k-1})$ as

$$p(\mathbf{x}_k|\mathbf{Z}_{k-1}) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{Z}_{k-1})d\mathbf{x}_{k-1}.$$

- $p(\mathbf{z}_k|\mathbf{x}_k)$ is defined by the measurement model and known statistics of $\mathbf{w}$

Introduction to Iterative Bayesian Methods: Continued

Knowledge of the posterior density provides a solution to form an estimate with respect to any reasonable criterion.

- As an example, the Minimum Mean Square Error (MMSE) estimate is given by

$$E\{\mathbf{x}_k|Z_k\} = \int \mathbf{x}_k \cdot p(\mathbf{x}_k|\mathbf{Z}_k) d\mathbf{x}_k$$

- Maximum Aposteori Probability (MAP) estimate is given by

$$\arg\max_{x_k} p(\mathbf{x}_k|\mathbf{Z}_k)$$

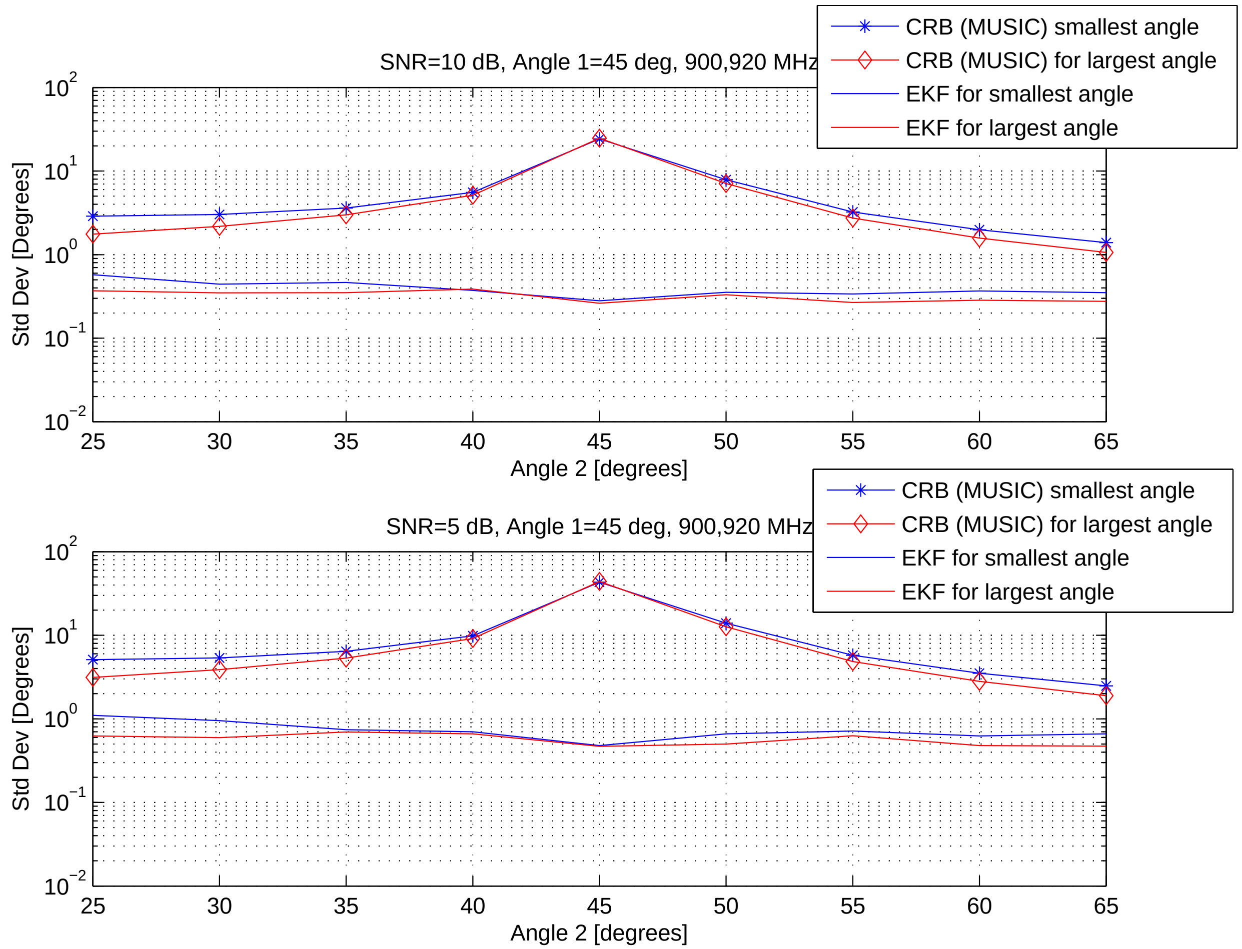
Low complexity approximation to Universal Bayesian Methods

- The Universal Bayesian solution has not been solved analytically to date (except for some trivial cases).
- Generally, we need approximations to obtain results in practical systems.
- The Extended Kalman Estimator (EKE) is a widely applicable approximation to the Universal formulation.
- For the EKE formulation we assume the first term in the Taylor series expansion of f and h are sufficient. This linearizes the non-linear Universal Bayesian solution.
- The posterior distribution $p(x_k|Z_k)$ is approximated as Gaussian.

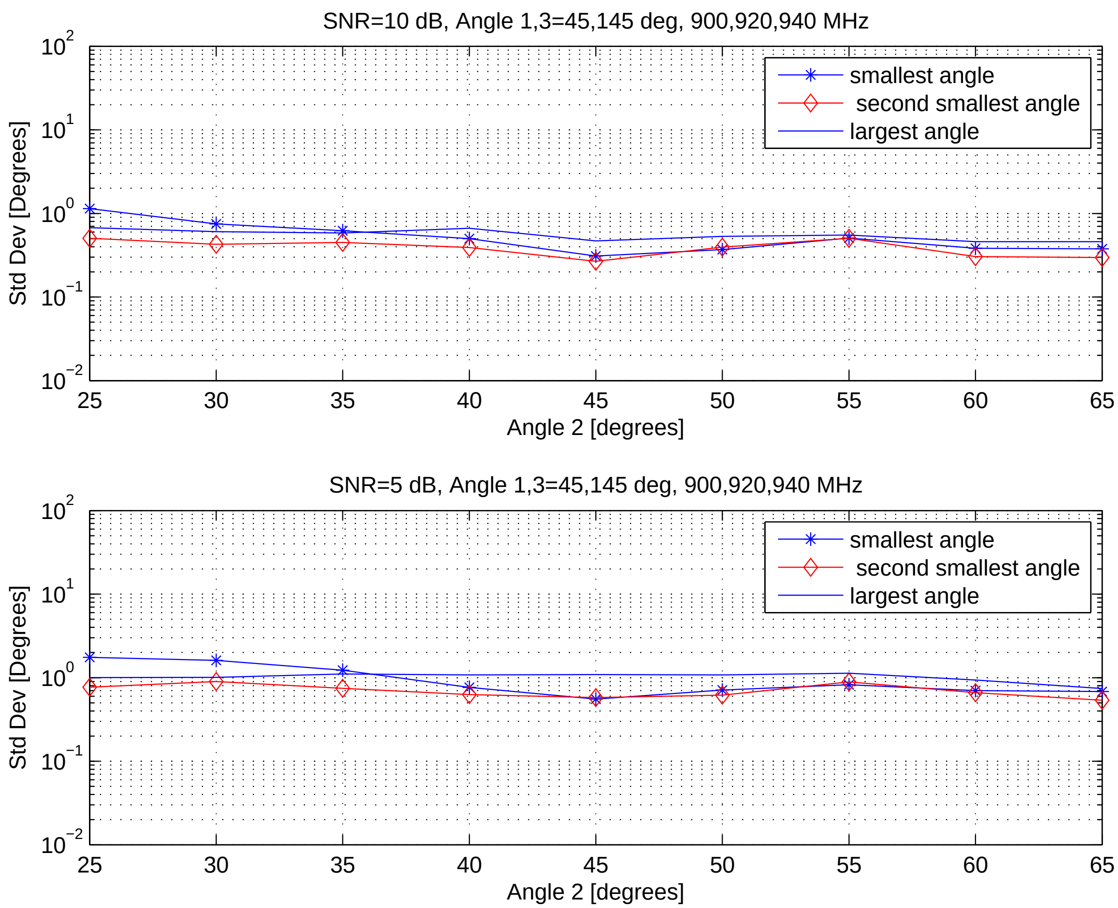
Numerical results: AoA estimation

- 200 samples of sinusoidal pulses
- Frequencies at 900 MHz, 920 MHz and 940 MHz
- All pulses have same amplitude
- No knowledge of number of pulses required as is the case with MUSIC

Numerical results: AoA estimation



Numerical results: AoA estimation



Conclusions

- AoA estimation can be performed effectively using oversampling.
- Limitations of MUSIC may be overcome to large extent.
- Synthesis of sparse array optimal for this formulation is open problem.
- There is no need to know the number of pulses, their amplitudes or phases, nor their frequencies.